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SHAKEDOWN OF UNIDIRECTIONAL COMPOSITES

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Abstract—The shakedown problem for a composite lamina made of an elastic-plastic matrix and elastic cylindrical fibers is studied. The plastic deformation modes of the lamina are reviewed, and it is concluded that significant shakedown effects can be caused only by the $I_1 = 1/2(T_{11} + T_{22})$ and $I_2 = T_{33}$ components of the remotely applied stress field which are symmetric about the axis x_3 of the fiber; T_{11} and T_{22} are the normal composite stresses in the transverse plane. It is shown that the I_1I_2 stress system is needed also to represent thermal loads caused by a uniform change of temperature in the composite.

Two methods for evaluation of shakedown limits in the I_1I_2-plane are described. First, the classical approach involving the determination of parametric families of self-stress fields and the solution of mathematical programming problems is used. Results are presented for selected *B–Al*, *Be–Al*, *B–Ti* and *B–Mg* composites.

In the second method, the shakedown problem is related to the recently developed kinematic hardening rules for fibrous composites. It is shown that the composite will shake down for any loading program within a prescribed domain in the I_1I_2-plane, providing that the domain can be contained within a translated initial yield surface. This approach leads to a closed-form evaluation of shakedown limits for any arbitrary combination of mechanical and thermal variable cyclic loads in fibrous composites with temperature-dependent matrix yield strengths.

The relationship between shakedown and fatigue in metal matrix composites is discussed.

NOTATION

$[A]$	coefficient matrix of the elastic microstresses
a	radius of the cylindrical fiber
b	radius of the composite cylinder
$\mathbf{c}, \mathbf{c}_p$	assembled and element coefficient matrix associated with the yield criterion, respectively
b_i $(i = 1, 2, 3, 4)$	constants defining the loading program
E_f, E_m	Young's modulus of the fiber and matrix material, respectively
F_i $(i = 1, 2, \ldots 10)$	dimensionless redundants defined by equation (19)
G_f, G_m	shear modulus of the fiber and matrix material, respectively
I_i $(i = 1, 2, \ldots 5)$	stress invariants
k	load factor
K_f, K_m	bulk modulus of the fiber and matrix material, respectively
$\mathbf{m}_p^E$	local maximum elastic stress response vector in element p
$\mathbf{N}, \mathbf{N}_f, \mathbf{N}_p$	assembled, fiber and element coefficient matrix associated with the yield criterion, respectively
$\mathbf{q}$	vector consisting of all redundants of the structure
$\mathbf{R}, \mathbf{R}_p$	assembled, and element matrix relating $\mathbf{q}$ to $\boldsymbol{\sigma}$ defined by equation (6), respectively
$r_1, \Delta r_1$	internal radius and thickness of the nested thick-walled tubes
r, θ, z	local cylindrical coordinates
$\mathbf{s}$	deviatoric component of microstress
$\mathbf{s}_r$	deviatoric component of selfstress
T_{ij} $(i,j = 1, 2, 3)$	macroscopic or composite stresses in a Cartesian coordinate system
V_f	fiber volume fraction
$\mathbf{x}$	vector with nonnegative components
x_i $(i = 1, 2, 3)$	Cartesian coordinate system
Y, Y_f	tensile yield stress of the matrix, and the fiber, respectively
α_f, α_m	coefficients of thermal expansion in the fiber and matrix, respectively
$\boldsymbol{\beta}$	vector consisting of ratios of the fiber and matrix tensile yield stresses
$\Delta\theta$	uniform temperature change (deg. F)
θ	temperature (deg. F)

λ factor of safety with respect to shakedown
ν_f, ν_m Poisson's ratio of fiber and matrix material, respectively
ϵ_{zz} uniform normal strain in z-direction
$d\mu$ scalar multiplier
$\boldsymbol{\sigma}$ total microstresses
$\boldsymbol{\sigma}^e$ elastic microstresses
$\boldsymbol{\sigma}^r$ local selfstress state
σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz} microstresses in cylindrical coordinates
κ coefficient describing the translation of initial yield surface on the I_1I_2-plane

INTRODUCTION

It is well known that elastic-plastic bodies or structures subjected to variable repeated loads in the plastic range can experience failure by cyclic plastic deformations or, alternatively, can shake down. According to the first shakedown theorem or Melan's theorem[13], the body will shake down for an arbitrary loading program within prescribed limits if any time-independent selfstress state can be found such that superposition of this state and the elastic response for all possible combinations of external forces and temperature within the prescribed limits will not lead to stresses at or above yield at any point[8, 15, 18]. If the body shakes down, it is safe against plastic failure but not necessarily against other failure modes, such as fatigue.

It is self-evident that the initial yield surface, or any subsequent loading surface, represents a lower bound on the shakedown limits of the structure, and that the shakedown load may not exceed the plastic limit load. Therefore, highest shakedown limits can be expected for such loading conditions where the plastic limit load is considerably greater than the initial yield load. The shakedown effect is absent when the initial yield and limit loads coincide.

This work is concerned with the determination of shakedown limits of unidirectional fibrous composites consisting of continuous, strong elastic fibers embedded in an elastic-perfectly plastic matrix. In most fibrous composites, and especially in those reinforced by brittle fibers, the ultimate strength of the fiber is considerably larger than the yield strength of the matrix. For example, in as-fabricated aluminum-boron composites the fiber tensile strength is equal to about forty times the uniaxial matrix yield stress. It follows that the highest plastic limit load of the composite is obtained for such loading modes where the plastic collapse of the matrix is controlled by the fiber strength. Those are the axisymmetric modes consisting of combinations of the normal stress in the fiber direction and the lateral hydrostatic stress in the transverse plane. Since one would expect the most pronounced shakedown effects to occur under these stress states, we shall limit our attention to the axisymmetric loading of the composite. The axisymmetric states include also a uniform thermal change, and are very frequently applied in practice.

This is not to suggest that shakedown effects are absent for other stress states which include the transverse and/or longitudinal shear of the composite lamina. However, it has been shown by several investigators, e.g. Shu and Rosen[17], McLaughlin[12], Lance and Robinson[10], and Butler and Sullivan[1], that the strength of the composite for these two shear stress states is not greatly affected by the presence of the fibers, except at very high fiber volume fractions which are seldom seen in practice. Furthermore, Dvorak *et al.* [2, 3] have found that the initial yield surfaces of unidirectional composites loaded in longitudinal and/or transverse shear nearly coincide with the yield surfaces of the matrix. Therefore, the composite may not shake down under the purely shear loads. Relatively small shakedown effects are expected for combinations of the axisymmetric and shear stress states.

The method of analysis used herein is similar to the finite element, linear programming approach described by Maier[11]. However, it will be shown that the shakedown limits for the axisymmetric loading of a composite lamina can also be obtained by a much simpler procedure which is based on a recently developed plasticity theory of fibrous composites (Dvorak and Rao[5, 16]).

THE MATERIAL MODEL

The unidirectional fibrous composite can be regarded as a transversely isotropic, macroscopically homogeneous solid consisting of an elastic-perfectly plastic matrix, and elastic fibers. A perfect bond is assumed to exist between the constituents. On the microscale, the fibers are right circular cylinders with parallel axes. The fibers can have different diameters and can be randomly distributed, however, the fiber volume fraction V_f must be statistically uniform in the transverse plane.

Suppose that we select a Cartesian coordinate system x_i such that the axes x_1, x_2 are in the transverse plane, and x_3 is parallel to the fiber direction. The macroscopic or composite stresses in this coordinate system are denoted as T_{ij} $(i,j = 1,2,3)$. If the composite is initially free of internal stresses, the local microstresses and the macroscopic strains in the composite caused by the composite stresses T_{ij} must be invariant under rotations about the x_3 axis, and under the transformation $x_3' = -x_3$. The invariants of the stress tensor T_{ij} which reflect these requirements can be obtained by analogy with the well-known strain invariants (e.g. Green and Zerna[6], p. 160). In the notation used in the earlier studies[5, 14], the stress invariants are:

$$\begin{aligned} I_1 &= \frac{1}{2}(T_{11}+T_{22}), \quad I_2 = T_{33}, \\ I_3 &= T_{13}^2 + T_{23}^2, \quad I_4 = \frac{1}{2}(T_{11}-T_{22})^2 + 2T_{12}^2 \\ I_5 &= \frac{1}{2}(T_{11}-T_{22})(T_{13}^2 - T_{23}^2) + 2T_{12}T_{23}T_{31}. \end{aligned} \tag{1}$$

As stated in the Introduction, we shall concern ourselves only with the axisymmetric composite stress states which are described by the first two invariants. Therefore, for the purpose of the present study we shall assume that $I_3 = I_4 = I_5 \equiv 0$.

The composite stress states caused by I_1 and I_2 are symmetric about the axis of each fiber. One can then assume that the local microstress states will be also symmetric about the fiber axes, both in the fiber, and in the matrix next to the fiber-matrix interface. Accordingly, as discussed by Hashin[7], the composite microstructure can be modelled as a system of right circular cylinders consisting of fibers surrounded by uniform layers of the matrix material. If the fiber volume fraction V_f is uniform in the transverse plane, it is sufficient, for analytical purposes, to consider a single composite cylinder of external radius b, containing a concentric cylindrical fiber of radius a; $V_f = (a/b)^2$, as shown in the left part of Fig. 1. Here we have introduced local cylindrical coordinates r, θ, z, such that z coincides with the cylinder axis, and r, θ are in the transverse plane. Under any combination of I_1, I_2, the cylinder is in the state of generalized plane strain in the $r\theta$-plane.

THE SHAKEDOWN ANALYSIS

The microstresses $\boldsymbol{\sigma}^e$ caused in the elastic composite cylinder by the composite stresses I_1, I_2 can be written as[5]:

$$\begin{aligned} \{\sigma_{rr}\sigma_{\theta\theta}\sigma_{zz}\} &= [A]\{I_1 I_2\}, \\ \{\sigma_{r\theta}\sigma_{\theta z}\sigma_{zr}\} &= 0, \end{aligned} \tag{2}$$

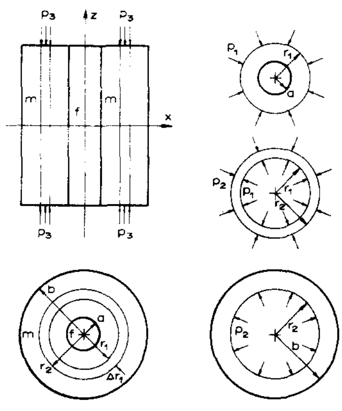

Fig. 1. Generation of selfstress states in the composite cylinder model.

where the braces $\{\} \equiv [\,]^T$ denote a column vector, or a transposition of a row matrix. The coefficients of the matrix $[A]$ must be evaluated as a function of the radius r; they depend also on the four elastic constants of the constituents, and on the fiber volume fraction V_f.

Let $\boldsymbol{\sigma}^r$ denote the local selfstress state at any given stage of a loading program in the I_1I_2-plane. Then, the total microstresses can be written as

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^e + \boldsymbol{\sigma}^r. \tag{3}$$

As indicated earlier, we shall assume that yielding will be limited to the matrix. The case of possible yielding in both fiber and matrix can be treated in a similar manner, and will be discussed in the sequel. Suppose that the yield condition in the matrix is expressed by a set of linear inequalities

$$f(\boldsymbol{\sigma}) = \mathbf{N}^T\boldsymbol{\sigma} - \mathbf{c} \le 0, \tag{4}$$

where $\mathbf{N}$ and $\mathbf{c}$ are coefficient matrices associated with the yield criterion.

The first shakedown theorem stipulates that shakedown to some selfstress will take place if any selfstress $\boldsymbol{\sigma}^r$ can be found such that at every point and instant:

$$\begin{aligned} &f(k\boldsymbol{\sigma}^e + \boldsymbol{\sigma}^r) \le 0, \text{ i.e.} \\ &k\mathbf{N}^T\boldsymbol{\sigma}^e + \mathbf{N}^T\boldsymbol{\sigma}^r \le \mathbf{c}. \end{aligned} \tag{5}$$

Here $k \ge 0$ is a load factor such that the structure does shake down for any $k \le \lambda$, and does not shake down for $k > \lambda$; λ is the factor of safety with respect to shakedown.

Suppose that the matrix domain is divided into n concentric annular rings such as shown in Fig. 1, and let $\mathbf{q}$ represent a vector consisting of all redundants of the structure. Then, the selfstress vector in each element p can be written as

$$\boldsymbol{\sigma}_p{}^r = \mathbf{R}_p\mathbf{q}, \quad p = 1, 2, \ldots n, \tag{6}$$

where the matrix $\mathbf{R}_p$ depends on the choice of $\mathbf{q}$, and on the geometry and elastic constants of the elements of the structure. Let the vector

$$\mathbf{m}_p{}^E = \max(\mathbf{N}_p{}^T\boldsymbol{\sigma}_p{}^e) \tag{7}$$

represent the local maximum elastic stress response in the element p, for all possible variations of the external loads within the prescribed limits.

Then, the problem of finding a maximum value of the load factor $k \leq \lambda$ which will cause shakedown can be formulated as a linear programming problem:

Maximize the load factor $k \geq 0$, the objective function of the variables $\mathbf{q}$, subject to the constraints

$$k\mathbf{m}_p^E + \mathbf{N}_p^T\mathbf{R}_p\mathbf{q} \leq \mathbf{c}_p, \quad p = 1, 2, \ldots n. \tag{8}$$

When the matrices $\mathbf{N}_p$, $\mathbf{m}_p^E$, $\mathbf{R}_p$, and $\mathbf{c}_p$ are assembled for each element of the structure, the shakedown problem can be formulated in a compact form:

$$\lambda = \max\{k | k\mathbf{m}^E + \mathbf{N}^T\mathbf{R}\mathbf{q} \leq \mathbf{c}, \quad k \geq 0\} \tag{9}$$

Alternatively, one can reformulate equation (9) as a dual problem in linear programming:

$$\lambda = \min\{\mathbf{c}^T\mathbf{x} | (\mathbf{m}^E)^T\mathbf{x} = 1, \mathbf{R}^T\mathbf{N}\mathbf{x} = \mathbf{0}, \mathbf{x} \geq \mathbf{0}\} \tag{10}$$

where $\mathbf{x}$ is a new variable vector with nonnegative components.

The generation of a suitable parametric family of self-stress fields is of essence in the shakedown analysis. To this end, the matrix domain of the composite cylinder can be divided into a set of nested thick-walled tubes of internal radius r, and wall thickness Δr_1 (Fig. 1). Independent uniform stresses p_1, p_2 and p_3, applied to each of the elements cause the following stress distribution in the composite cylinder.

For $a \leq r \leq r_1$:

$$\begin{aligned}
\sigma_{rr}^{(1)} &= -p_1 + (p_a - p_1)(1 - r_1^2/r^2)[a^2/(r_1^2 - a^2)] \\
\sigma_{\theta\theta}^{(1)} &= -p_1 + (p_a - p_1)(1 + r_1^2/r^2)[a^2/(r_1^2 - a^2)] \\
\sigma_{zz}^{(1)} &= -2\nu_m p_1 + 2\nu_m(p_a - p_1)[a^2/(r_1^2 - a^2)] + E_m\epsilon_{zz},
\end{aligned} \tag{11}$$

where ϵ_{zz} is the uniform normal strain in the z-direction, and the interface stress:

$$p_a = 2G_f\frac{p_1(1-\nu_m) + G_m(1 - a^2/r_1^2)(\nu_m - \nu_f)\epsilon_{zz}}{G_m(1 - a^2/r_1^2)(1 - 2\nu_f) + G_f(1 - 2\nu_m)a^2/r_1^2 + G_f}.$$

E_m, G_m, ν_m, E_f, G_f, ν_f are the elastic constants of the matrix and fiber, respectively.

For $r_1 \leq r \leq r_1 + \Delta r_1 = r_2$:

$$\begin{aligned}
\sigma_{rr}^{(2)} &= -p_2 + (p_1 - p_2)(1 - r_2^2/r^2)[r_1^2/(r_2^2 - r_1^2)] \\
\sigma_{\theta\theta}^{(2)} &= -p_2 + (p_1 - p_2)(1 + r_2^2/r^2)[r_1^2/(r_2^2 - r_1^2)] \\
\sigma_{zz}^{(2)} &= -p_3 + E_m\epsilon_{zz}.
\end{aligned} \tag{12}$$

For $r_2 \leq r \leq b$:

$$\begin{aligned}
\sigma_{rr}^{(3)} &= p_2(1 - b^2/r_2^2)[r^2/(b^2 - r_2^2)] \\
\sigma_{\theta\theta}^{(3)} &= p_2(1 + b^2/r^2)[r_2^2/(b^2 - r_2^2)] \\
\sigma_{zz}^{(3)} &= 2\nu_m p_2[r_2^2/(b^2 - r_2^2)] + E_m\epsilon_{zz}.
\end{aligned} \tag{13}$$

Equilibrium of the selfstresses in the z-direction requires:

$$0 = \frac{1}{A}(\sigma_{zz}^{(f)}A_f + \sigma_{zz}^{(1)}A_m^{(1)} + \sigma_{zz}^{(2)}A_m^{(2)} + \sigma_{zz}^{(3)}A_m^{(3)}), \tag{14}$$

where A_f is the cross-sectional area of the fiber, $\sigma_{zz}^{(f)} = -2\nu_f p_a + E_f \epsilon_{zz}$, and $A_m^{(i)}$, $(i = 1, 2, 3)$ are the cross-sectional areas of the respective parts of the matrix.

In the present solution, the matrix domain was divided into five elements of the type shown in Fig. 1, such that their thickness was uniform, $\Delta r_1 = (b - a)/5$. Under such circumstances one can find three independent stresses p_1, p_2, p_3 in each internal element, and two stresses p_2, p_3 for the fifth element $a + 4\Delta r_1 \leq r \leq b$. Equilibrium at the interfaces leads to a reduction in the number of independent internal stresses which generate the parametric family of selfstress states.

One can write:

$$\begin{aligned} &\text{at } r_1 = a:\ p_1^{(1)} = q_1, p_3^{(1)} = q_2; \\ &\text{at } r_1 = a + \Delta r_1:\ p_2^{(1)} + p_1^{(2)} = q_3, p_3^{(2)} = q_4; \\ &\quad \cdots\cdot \\ &\text{at } r_1 = a + 4\Delta r_1:\ p_2^{(4)} + p_1^{(5)} = q_3, p_3^{(5)} = q_{10}. \end{aligned} \tag{15}$$

Therefore, the selfstress state in the five-element matrix domain can be generated in terms of ten independent stresses $q_1, q_2, \ldots q_{10}$. Specifically, one can find from equations (11)–(15) the selfstress in element p as:

$$\begin{Bmatrix} \sigma_{rr}^r \\ \\ \sigma_{\theta\theta}^r \\ \\ \sigma_{zz}^r \end{Bmatrix}_p = \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1,10} \\ \\ R_{21} & R_{22} & \cdots & R_{2,10} \\ \\ R_{31} & R_{32} & \cdots & R_{3,10} \end{bmatrix}_p \begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_{10} \end{Bmatrix}, \tag{16}$$

$p = 1, 2, \ldots n$; which is, of course, an explicit form of equation (6).

The determination of shakedown limits was based on the Tresca yield condition

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{Bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \end{Bmatrix} \leq Y \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}, \tag{17}$$

as in equation (4). Y is the tensile yield stress of the matrix. The explicit form of the shakedown condition, equation (8), for an element p follows from equations (7, 8, 16 and 17). The determination of the vector $\mathbf{m}_p^E$ in equation (7) is contingent upon the specification of the load limits; the elastic stresses $\boldsymbol{\sigma}_p^e$ can be evaluated by obvious modifications of equation (11).

The maximum load factor k was determined by solution of the linear programming problem, equation (9), which was written for a total of eleven points located at the boundaries, and at the centers of each of the five elements. Specifically, the points were located at $r = a$, $r = a + 1/2\Delta r_1$, $r = a + \Delta r, \ldots r = b$, where again, $\Delta r_1 = (b - a)/5$. The explicit form of equation (8) was:

$$\underset{(66\times 1)}{\frac{k}{Y}\begin{Bmatrix} \mathbf{m}_1^E \\ \mathbf{m}_2^E \\ \vdots \\ \mathbf{m}_{11}^E \end{Bmatrix}} + \underset{(66\times 33)}{\begin{bmatrix} \mathbf{N}_1^T & & & 0 \\ & \mathbf{N}_2^T & & \\ & & \ddots & \\ 0 & & & \mathbf{N}_{11}^T \end{bmatrix}} \underset{(33\times 10)}{\begin{bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \vdots \\ \mathbf{R}_{11} \end{bmatrix}} \underset{(10\times 1)}{\begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_{10} \end{Bmatrix}} \leq \underset{(66\times 1)}{\begin{Bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{Bmatrix}}, \tag{18}$$

where F_i are new dimensionless redundants,

$$F_i = \frac{q_i}{Y}, \quad i = 1, 2, \ldots 10. \tag{19}$$

The solution was constructed by a numerical procedure based on the simplex method, following the approach of Künzi *et al.* [9]. The dual minimization problem, equation (10), was solved. The results are equivalent to those of the primal problem, the advantage of the dual problem is that no nonnegativity restrictions need to be imposed on F_i.

RESULTS FOR DIFFERENT LOADING PROGRAMS

The shakedown limits were evaluated for a number of different loading regimes, and for several types of metal matrix composites. In these calculations, it was assumed that there is no Bauschinger effect in the matrix, and that the material is free of initial stresses. The latter assumption can be made without the loss of generality, since the magnitude of shakedown limits of a structure does not depend upon the existence of initial residual stresses. Because of the radial symmetry of the composite cylinder, and the applied load in the I_1I_2-plane, it was necessary to consider only one-half of the loading plane, and that was taken as $I_2 \geq 0$.

The results obtained for different loading programs will now be described.

(a) Proportional loading $I_2 = \alpha I_1$, $-b \leq I_1/Y \leq b$. It is obvious that this symmetric radial loading program must give shakedown limits which are identical with the initial yield limits. Therefore, the shakedown limits surface must be identical with the initial yield surface of the composite [2, 3]. This was confirmed by the present calculations. The initial yield surface in the I_1I_2-plane is illustrated in Fig. 2 for the case of a boron-aluminum composite, $V_f = 0{\cdot}68$. The segments $F_1'H_1'$, $F_2'H_2'$, and $F_3'H_3'$ represent the shakedown limits for the three different radial directions. The initial yield ellipse is the locus of the shakedown limits for any proportional path.

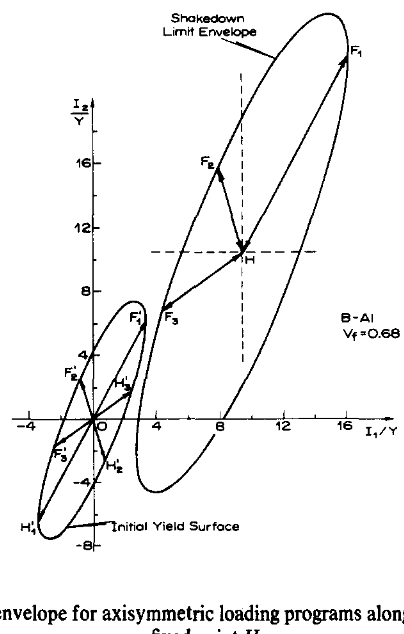

Fig. 2. The shakedown limit envelope for axisymmetric loading programs along a rectilinear path HF_i from a fixed point H.

(b) Loading along a rectilinear path $I_2 = \alpha I_1$, $b_1 \le I_1/Y \le b_2$. The results are shown in Fig. 2. The point H is fixed, it corresponds to one of the limits b_1, or b_2. The envelope of the shakedown limits on any path HF_i, $i = 1, 2, 3, \ldots$, is an ellipse which is identical with the initial yield surface magnified by a factor of two. It is readily seen that each of the segments HF_1, HF_2, and HF_3 has the same length as the respective parallel segments $H_1'F_1'$, etc. through the origin.

(c) General variable loading within a rectangular domain $b_1 \le I_1/Y \le b_2$, $b_3 \le I_2/Y \le b_4$. The results are shown in Fig. 3. Again, the point H is assumed to be fixed; its coordinates provide two of the four bounds b_j. The result of the calculation is the position of the point F_i ($i = 1, 2, 3, 4, \ldots$), which provides the remaining two bounds. The solid diamond-shaped contour $ABCD$ was found to be the localle or envelope of the points F_i of the loading domains $E_iF_iG_iH$, generated at the fixed point H, which satisfy the shakedown condition, equation (18), for the composite cylinder. As in Fig. 2, the position of H could be arbitrary, the admissible load limits would merely translate in the I_1I_2-plane.

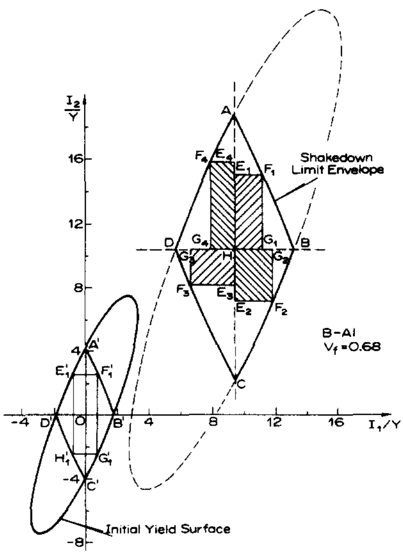

Fig. 3. The shakedown limit envelope for axisymmetric loading programs within rectangular domains $E_iF_iG_iH$. The point H is fixed.

It is observed that the contour $ABCD$ can be constructed from the magnified initial yield surface shown in Fig. 2, which is redrawn in a dashed line in Fig. 3. Therefore, it can be easily established that each of the rectangular domains $E_iF_iG_iH$, when translated so that their centers coincide with the origin O, would exactly fit into the initial yield surface. Fig. 3 shows one of such translated domains $E_1'F_1'G_1'H_1'$. The contour $A'B'C'D'$ is a one-half reduction of $ABCD$, and is the localle or envelope of the corners of the translated rectangular loading domains.

RESULTS FOR DIFFERENT COMPOSITE SYSTEMS

Results similar to those shown in Figs. 2 and 3 were obtained also for several other composite systems, and for different volume fractions of the fiber. These results showed the same general features mentioned before. Specifically, one can construct the shakedown limits from the initial yield surface of the composite in the I_1I_2-plane. The equations of the initial yield surfaces for several selected composites are summarized in Table 1. These were taken from earlier work [2, 3]

Table 1. Coefficients of equations of initial yield surfaces, and thermal shift coefficients κ. $f \equiv K_1I_1^2 + K_2I_1I_2 + K_3I_2^2 - Y^2 = 0$ (equation 20); $\Delta P = \Delta I_1 = \Delta I_2 = \kappa\,\Delta\theta$ (equation 30)

Material	V_f	K_1	K_2	K_3	κ (psi/°F)
	0·3	0·4738	−0·5173	0·1798	
B–Al	0·5	0·3549	−0·3110	0·0929	−377
	0·68	0·2951	−0·2207	0·0590	
	0·3	0·4072	−0·6986	0·3062	
Be–Al	0·5	0·2664	−0·4255	0·1755	−758
	0·68	0·2000	−0·3004	0·1178	
	0·3	0·4286	−0·3189	0·0876	
B–Mg	0·5	0·3337	−0·1820	0·0402	−252
	0·68	0·2851	−0·1265	0·0241	
	0·3	0·4793	−0·7065	0·2890	
B–Ti	0·5	0·3451	−0·4477	0·1672	−45
	0·68	0·2772	−0·3249	0·1129	

Elastic constants of the constituents

Material	E (10^6 psi)	G (10^6 psi)	ν	α (10^{-6}/°F)
Al	10·5	3·95	0·3291	13·0
B	58·0	23·97	0·2098	4·5
Be	40·0	19·60	0·0204	6·5
Mg	6·5	2·44	0·3319	15·0
Ti	15·0	5·59	0·3417	5·0

and are based on the Mises rather than the Tresca yield condition. However, the differences in the shapes of the surfaces due to the choice of the yield condition are very minor.

Figure 4 shows the magnitudes of shakedown loads for the case of pure tension in the fiber direction ($I_1 = 0$), for $R = (I_2)_{min}/(I_2)_{max} = 0$. The lines in the graph represent the shakedown limit $(I_2)_{max}$ for the zero-tension loading cycle. It is seen that the limit increases linearly with the fiber volume fraction, and most rapidly so for the boron-magnesium system. Note that all lines would intersect at $I_2/Y = 2$ for $V_f \to 0$, which suggests that significant shakedown effects are present even at low volume fractions of the fiber.

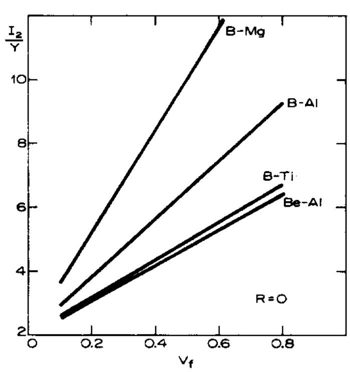

Fig. 4. Shakedown limits for four composite systems loaded in the zero-tension mode in the fiber direction.

RELATIONSHIP TO THE HARDENING RULES

In a recent study, Dvorak and Rao[5, 16] have formulated new kinematic hardening rules for the axisymmetric deformation of unidirectional fibrous composites. The rules have been verified by comparison with the exact finite element plasticity analysis, and are considered to be very reliable and accurate. Specifically, the hardening rules indicate that if, in the I_1I_2-plane, the initial yield surface of the composite is of the form

$$f(I_1, I_2) = 0, \tag{20}$$

then any subsequent loading surface can be expressed in the form

$$f(I_1 - \alpha_1, I_2 - \alpha_2) = 0. \tag{21}$$

It follows that the loading surfaces are identical with the initial yield surface which has experienced a rigid body translation in the I_1I_2-plane.

The magnitude of the translation coefficients must be determined by integration along the loading path of the equations of the type

$$\begin{Bmatrix} \mathrm{d}\alpha_1 \\ \mathrm{d}\alpha_2 \end{Bmatrix} = \begin{Bmatrix} \mathrm{d}I_1 \\ \mathrm{d}I_2 \end{Bmatrix} - \mathrm{d}\mu \begin{Bmatrix} -\partial f/\partial I_2 \\ \partial f/\partial I_1 \end{Bmatrix}, \tag{22}$$

where $\mathrm{d}\mu$ is a scalar multiplier.

The results shown in Figs. 2 and 3 suggest that any loading surface is an envelope of loading programs for which shakedown of the composite will occur. Indeed, that is a self-evident consequence of equation (21), and of the Melan's theorem, since each loading surface corresponds to a selfstress state which assures elastic deformation for all loads within that surface. Conversely, it can be shown that shakedown is possible only for those loading programs that can be contained within a loading surface of the type described by equation (21). Suppose that $\mathbf{s}_0$ is a unit deviatoric stress vector at the matrix point where yielding first starts when a stress-free composite is loaded by a proportional axisymmetric composite stress state $\mathbf{I}$. Specifically, the deviatoric stress $\mathbf{s}$ at this point is, in analogy with equation (2),

$$\mathbf{s} = \beta \mathbf{s}_0 = \mathbf{PI} \tag{23}$$

where β is a scalar multiplier, and the matrix $\mathbf{P}$ is related to $\mathbf{A}$ of equation (2), (see, equations (7–11) in [5]). The magnitudes of β at yield are obtained from the yield condition

$$\beta^2 \mathbf{s}_0^T \mathbf{s}_0 = k^2, \tag{24}$$

i.e.

$$\beta_1 = -\beta_2 = k(\mathbf{s}_0^T \mathbf{s}_0)^{-1/2} = \beta_0. \tag{25}$$

Therefore, for any radial vector $\mathbf{s} = \beta \mathbf{s}_0$,

$$\beta_1 - \beta_2 = 2\beta_0. \tag{26}$$

Consider next that a selfstress state exists in the composite and that a composite stress program is applied within a certain segment of a proportional path coinciding with $\mathbf{I}$. Yielding may now start at a different point of the matrix. However, at the onset of yielding, the deviatoric stresses at some point of the matrix must satisfy the yield condition, equation (24), in the form

$$(\mathbf{s}_r + \beta \mathbf{s}_0)^T (\mathbf{s}_r + \beta \mathbf{s}_0) = k^2, \tag{27}$$

where $\mathbf{s}_r$ is the selfstress, and $\mathbf{s}_0$ is the unit deviator stress, as in equation (23). Note that $\mathbf{P}$ may now correspond to a different initial yield point in the matrix. The solution of equation (27) can be obtained as

$$\beta_1 - \beta_2 = \frac{2[(\mathbf{s}_r{}^T\mathbf{s}_0)^2 - (\mathbf{s}_0{}^T\mathbf{s}_0)(\mathbf{s}_r{}^T\mathbf{s}_r) + k^2\mathbf{s}_0{}^T\mathbf{s}_0]^{1/2}}{\mathbf{s}_0{}^T\mathbf{s}_0}. \tag{28}$$

From the Schwarz inequality:

$$(\mathbf{s}_r{}^T\mathbf{s}_0)^2 \le (\mathbf{s}_0{}^T\mathbf{s}_0)(\mathbf{s}_r{}^T\mathbf{s}_r).$$

Therefore, if the solution of equation (28) exists, then

$$|\beta_1 - \beta_2| \le 2\beta_0, \tag{29}$$

where β_0 is given by equation (25).

Since **I** can assume any direction in the composite stress space, it follows from equations (25) and (29) that any shakedown limit surface in the composite stress space may not be larger than a translated initial yield surface. Therefore, the loading surface, equation (21), is also the largest possible shakedown limit surface and only those loading programs that can be contained within such a loading surface will cause shakedown. Q.E.D.

Figure 5 represents schematically the relationship between the initial yield and shakedown limit surfaces. The composite will shake down for any loading within an arbitrary domain $ABCDE$, providing that a congruent domain $A'B'C'D'E'$, translated in the I_1I_2-plane can be contained within the initial yield surface of the composite.

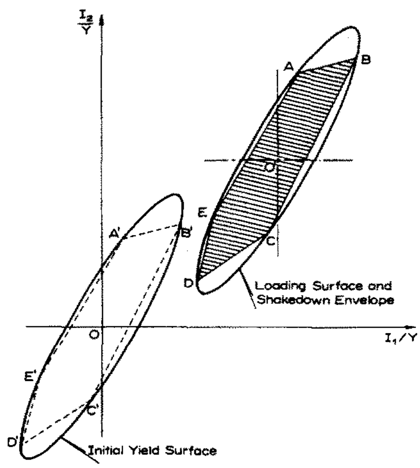

Fig. 5. The shakedown limit envelope for axisymmetric loading programs within an arbitrary domain *ABCDE*.

SIMULTANEOUS MECHANICAL AND THERMAL LOADS

In many practical applications the composites are exposed to simultaneous mechanical and thermal variable cyclic loads. If the magnitude of the yield stress of the matrix is not affected by the thermal change, the shakedown limits for a combined thermomechanical loading program can be found by the previously described procedure. Specifically, equations (9) or (10) can be used to formulate the linear programming problem. The parametric family of selfstress states is independent of the type of loading. The thermal load will affect only the magnitudes of the components of the vector $\mathbf{m}_p{}^E$, equation (7), and the corresponding terms in equations (8–10).

An alternative approach to the shakedown problem for both mechanical and thermal loads can be based on the hardening rules, equations (20–22). It was shown in Refs. [2] and [3] that in the absence of mechanical loads, a uniform thermal change in the composite leads to a rigid body translation of the initial yield surface in the direction of the hydrostatic axis. The same is true for any loading surface given by equation (21). The magnitude of the translation ΔP was shown to be (equation (16) in [3]), in the present notation:

$$\Delta P = \Delta I_1 = \Delta I_2 = 3\left[\frac{\alpha_m - \alpha_f}{\dfrac{1}{K_f} - \dfrac{1}{K_m}}\right]\Delta\theta = \kappa\Delta\theta, \tag{30}$$

where ΔI_1, ΔI_2 are the changes in the coordinates of the center of the initial yield surface caused by a uniform thermal change $\Delta\theta$; κ is the shift factor; α_m, α_f are the thermal expansion coefficients of the constituents, and K_m, K_f are their bulk moduli. Accordingly, if a variable cyclic thermal loading program is applied to the composite, the initial yield surface, equation (20), and Table 1, or a subsequent loading surface, equation (21), will experience a corresponding variable cyclic motion in the direction parallel to the hydrostatic axis $I_1/I_2 = 1$ in the I_1I_2-plane. The magnitudes of the shift factor are listed in Table 1 for selected composite systems. It can be easily established that relatively small thermal changes can cause yielding in most metal matrix systems—with the notable exception of the B–Ti composite—as discussed in [2] and [3].

The above considerations can be applied to the shakedown problem in an obvious way, which is illustrated schematically in Fig. 6. Suppose that an arbitrary loading program within prescribed limits $ABCD$ is combined with a thermal cycle. Let the two translated initial yield surfaces shown represent the extreme positions for the prescribed thermal cycle. Then, the composite will shake down if the domain $ABCD$, or any other prescribed load domain, can be contained within the area where the two initial yield surfaces overlap, as shown in Fig. 6. The same is true for any congruent loading domain $A'B'C'D'$ in the I_1I_2-plane, where translated loading surfaces, equation (21), replace now the initial yield surfaces.

The validity of equation (30) is limited, of course, only to the elastic deformation of the composite. In the absence of mechanical loads, the composite starts to deform plastically when

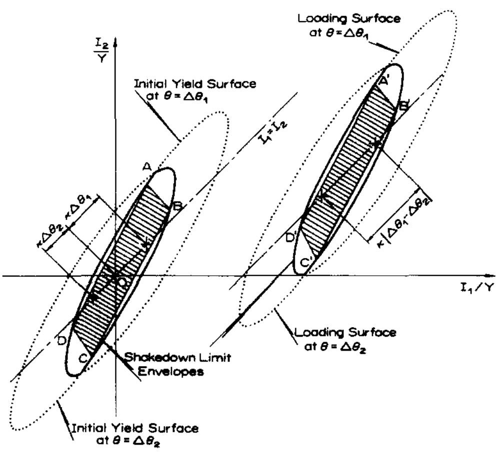

Fig. 6. Shakedown limit envelopes for simultaneous mechanical and thermal variable cyclic loads. Temperature variation interval: $\Delta\theta_1 \le 0 \le \Delta\theta_2$. Load domains: $ABCD$ or $A'B'C'D'$.

the translated yield surface touches the origin of coordinates in the I_1I_2-plane; the origin is the only common point of the translated yield surfaces in their extreme positions [3, 4]. It follows that the composite will not shake down under variable cyclic mechanical loads when the translated initial yield surfaces, or loading surfaces, have no common area of overlap at the extremes of the thermal cycle.

The direct approach shown in Fig. 6 is applicable even if Y is a function of temperature, providing that the size of the surfaces is adjusted at the extreme points of the thermal cycle.

DISCUSSION

The present procedures permit the evaluation of shakedown limits for axisymmetrically loaded composites. The more labourious approach consisting of the selection of the parametric families of self-stresses, and of the solution of linear programming problems can be replaced by a much simpler technique which permits a direct determination of the shakedown limits from the known shapes of the initial yield surfaces of the composites.

The principal application of the present results is, of course, in the prevention of fatigue in metal matrix composites. Specifically, it has been found in [4] that the shakedown limits coincide with the fatigue limit (or with fatigue strength at 10^6–10^7 cycles) for the same loading program in as-fabricated aluminum matrix composites, especially in the B–Al system. The reason for this relationship is seen in the fact that the yield stress of the aluminum matrix in the overaged condition is very nearly equal to the fatigue strength of the matrix after 10^6–10^7 cycles of symmetric tension-compression loading. This particular loading sequence is in fact experienced by the matrix in each shakedown state. Although the experimental confirmation is presently available only for uniaxial tests, both in the fiber and off-axis directions, it is felt that a similar relationship may exist for other stress states as well.

CONCLUSIONS

(a) Substantial shakedown effects in unidirectional metal matrix composites have been found for loading states which are axisymmetric with respect to the fiber axis, and which include also a uniform thermal change.

(b) The shakedown limits for a variable cyclic loading program, consisting of both mechanical and thermal loads, can be found by a standard procedure involving the selection of parametric families of selfstress states, and maximization of the allowable load factor by means of linear programming.

(c) An alternative, direct approach to the determination of the shakedown limits is related to the recently developed kinematic hardening rules for symmetrically loaded fibrous composites. Shakedown will occur if the loading domain can be contained within a translated initial yield surface in the axisymmetric stress plane. This approach yields itself readily to treatment of simultaneous thermal and mechanical loads, and permits the consideration of the temperature effects on the magnitude of the matrix yield stress during thermal load cycles.

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REFERENCES

1. T. W. Butler and E. J. Sullivan, Jr., On the transverse strength of fiber-reinforced materials. *J. Appl. Mech.* **40**, 523 (1973).
2. G. J. Dvorak, M. S. M. Rao and J. Q. Tarn, Yielding in unidirectional composites under external loads and temperature changes. *J. Comp. Materials* **7**, 194 (1973).
3. G. J. Dvorak, M. S. M. Rao and J. Q. Tarn, Generalized initial yield surfaces for unidirectional composites. *J. Appl. Mech.* **41**, 249 (1974).

4. G. J. Dvorak and J. Q. Tarn, Fatigue and shakedown in metal matrix composites. Duke University, Department of Civil Engineering, Solid Mechanics Series No. 7, 1973. To appear in ASTM Special Technical Publication No. 569.
5. G. J. Dvorak and M. S. M. Rao, Plasticity theory of fibrous composites—I. Duke University, Department of Civil Engineering, Solid Mechanics Series No. 8 (1973).
6. A. E. Green and W. Zerna, *Theoretical Elasticity*. Pergamon Press, Oxford (1953).
7. A. Hashin, The elastic moduli of heterogeneous materials. *J. Appl. Mech.* **29**, 143 (1962).
8. W. T. Koiter, A new general theorem on shakedown of elastic-plastic structures. *Proc. Koninkl. Nederl. Akad. van Wetenschappen*, Amsterdam, Series B59, No. 1, p. 24 (1956).
9. H. P. Künzi, H. G. Tzschach and C. A. Zehnder, *Numerical Methods of Mathematical Optimization: With Algol and Fortran Programs*. Academic Press, New York (1968).
10. R. H. Lance and D. N. Robinson, A maximum shear stress theory of plastic failure of fiber-reinforced materials. *J. Mech. Phys. Solids* **19**, 49 (1971).
11. G. Maier, Shakedown theory in perfect elastoplasticity with associated and nonassociated flow laws: a finite element, linear programming approach. *Meccanica* **4**, 250 (1969).
12. P. V. McLaughlin, Jr., Plastic limit behavior and failure of filament reinforced materials. *Int. J. Solids Structures* **8**, 1299 (1972).
13. E. Melan, Zur Plastizität des Räumlichen Kontinuums. *Ingenieur-Arch.* **9**, 116 (1938).
14. J. F. Mulhern, T. G. Rogers and A. J. M. Spencer, A continuum model for fiber-reinforced plastic materials. *Proc. R. Soc. London* **A301**, 473 (1967).
15. W. Prager, Shakedown in elastic, plastic media subjected to cycles of load and temperature. Brown University, Division of Applied Mathematics, Report No. DA-798/25 (May, 1956).
16. M. S. M. Rao and G. J. Dvorak, Plasticity theory of fibrous composites—II. Duke University, Department of Civil Engineering, Solid Mechanics Series No. 9 (1973).
17. L. S. Shu and B. W. Rosen, Strength of fiber-reinforced composites by limit analysis methods. *J. Comp. Materials* **1**, 366 (1967).
18. P. S. Symonds, Shakedown in continuous media. *J. Appl. Mech.* **18**, 85 (1951).